

# Analyzing the Accuracy of the Fitch Method for Reconstructing Ancestral States on Ultrametric Phylogenies

Louxin Zhang\*, Jian Shen†, Jialiang Yang‡, Guoliang Li§

June 17, 2009

## Abstract

Recurrence formulas are presented for studying the accuracy of the Fitch method for reconstructing the ancestral states in a given phylogenetic tree. As their applications, we analyze the convergence of the accuracy of reconstructing the root state in a complete binary tree of  $2^n$  as  $n$  goes to infinity and also give a lower bound on the accuracy of reconstructing the root state in an ultrametric tree.

**Keywords** Ancestral state reconstruction, analysis of reconstruction accuracy, Fitch method, phylogenetic trees.

## 1 Introduction

Ancestral sequence reconstruction incorporates sequences from modern living things into evolutionary models to estimate the corresponding sequence of an ancestor that died millions of years ago. This approach to understanding proteins was first suggested by Zuckerkandl and Pauling in their seminal work [7] in 1963. With the rapid accumulation of biomolecular sequence data and advances in computational biology, it has become an important approach to studying the origin and evolution of genes, proteins and even whole genomes (see for example [5] and [10]).

The Fitch method [3] was the first phylogenetic technique used for inferring the ancestral states of a character when the phylogeny that relates the ancestor to the extant species is known [1]. As a parsimony method, it estimates the ancestral state by minimizing the total number of hypothetical substitutions in all branches that are used to explain the evolution of the character states. It is efficient and accurate for sequences that are reasonably similar to each other. However, the accuracy of the Fitch method for reconstructing ancestral states has yet to be well studied [4, 6, 8, 11].

In this work, we present a set of recurrence formulas for analyzing the reconstruction accuracy of the Fitch method (in Theorem 3.1). These formulas are derived from a work of Maddison [6] (also see [9]).

---

\*Department of Mathematics, National University of Singapore (NUS), Singapore 117543. This work was supported by ARF(R146-000-109-112). Email: matzlx@nus.edu.sg

†Department of Mathematics, Texas State University, San Marcos, TX 78666, USA. This work was partially supported by NSF (CNS 0835834) and Texas Higher Education Coordinating Board (ARP 003615-0039-200). E-mail: js48@txstate.edu

‡MPI-CAS Institute of Computational Biology, CAS at Shanghai. Email: yangjialiang@picb.ac.cn

§Department of Computer Science, NUS. Email: ligl@gis.a-star.edu.sg

They are simple and useful as demonstrated in solving two theoretical problems that arise from studying the reconstruction accuracy of the Fitch method.

The first problem is to analyze the convergence of the accuracy of the Fitch method for reconstructing the root state in a complete phylogenetic tree in the equal-length branch and two-state Jukes-Cantor model (see Section 2 for details). Let  $p$  denote the conservation rate in each branch. In [9], Steel showed that, when the Fitch method is applied, the accuracy of reconstructing the root state from all leaf states in the complete binary tree of  $2^n$  leaves converges as  $n$  goes to infinity to  $\frac{1}{2}$  if  $\frac{1}{8} \leq p \leq \frac{7}{8}$  and  $\frac{1}{2} + \frac{1}{2} \frac{\sqrt{(8p-7)(4p-3)}}{(2p-1)^2}$  if  $\frac{7}{8} \leq p \leq 1$ . This result was proved under the assumption that suitable limits exists. However, the existence of these limits is not trivial. In this paper, we fill the gap left in [9] by proving that these limits exist. In addition, we also show that the reconstruction accuracy diverges when  $p \leq \frac{1}{8}$ .

Complete phylogenetic trees in which all branches have equal length are special ultrametric trees. In an ultrametric tree, each branch has its own branch length  $l(e)$ , with conservation rate  $p(e) = \frac{1}{2} (1 + e^{-l(e)})$  in the two-state Jukes-Cantor model, but requiring that the sum of branch lengths is constant in each path from the root to a leaf. A counterintuitive fact is that the reconstruction accuracy of the Fitch method is not a monotonic function of the size of taxa selected for reconstruction of the root state (even for ultrametric trees) [4]. Hence, Li et al asked whether the accuracy  $RA_F$  of the Fitch method for reconstructing the root state from all leaf states is always larger than or equal to the conservation rate along a root-to-leaf path or not in an ultrametric tree. Recently, this problem is positively answered by Fischer and Thatte [2]. In the second part of this paper, we present a stronger lower bound on  $RA_F$  for arbitrary ultrametric trees. Our bound implies that  $RA_F$  is not less than the accuracy of reconstructing the root state from any three leaves in an ultrametric tree.

## 2 The Fitch method and its reconstruction accuracy

Let  $C$  be a character with multiple states. Given a phylogenetic tree  $T$  of the character  $C$  in which each leaf has a state, the Fitch method estimates the root state from the leaf states in two steps. It first computes a subset  $S_u$  of states for each node  $u$  of  $T$  as follows:

1. If  $u$  is a leaf,  $S_u$  contains only the state of  $u$ ;
2. If  $u$  is an internal node having children  $v$  and  $w$ ,  $S_u$  is equal to  $S_v \cup S_w$  if  $S_v$  and  $S_w$  are disjoint and  $S_v \cap S_w$  otherwise.

After the subset  $S_r$  for the root of  $T$  is computed, the method selects a state as the root state from  $S_r$  randomly. In other words, a state is selected as the root state with probability  $\frac{1}{|S_r|}$ , where  $|S_r|$  denotes the number of states contained in  $S_r$ .

Assume the mutation process along each branch of the given tree is modeled as a stochastic process in which a state is replaced by another with some probability. The Fitch method reconstructs correctly a root state  $s$  from a set  $D$  of leaf states only if  $s$  evolves into the leaf states in  $D$ . Hence, the accuracy of the Fitch method for reconstructing the state of the root of  $T$ , denoted by  $RA_F(T)$ , is defined to be the expected probability that the Fitch method outputs a true state from a set  $D$  of leaf states. Let  $\Pr_r[D|s]$  denote the probability that the root state  $s$  evolves into the leaf states in  $D$ . Then,

$$RA_F(T) = \sum_{s,D} p_r(s) \Pr_r[D|s] \Pr[s \text{ is output from } D], \quad (1)$$

where  $p_r(s)$  is the prior probability of  $s$  being the root state.

### 3 Recurrence formulas for analyzing the reconstruction accuracy

In the rest of this paper, we assume that the character has only two states 0 and 1 and the root takes these two states with equal prior probability. By definition, the Fitch method selects 1 with probability 1 if  $\{1\}$  is the state subset  $S_r(D)$  computed from  $D$  at the root in the first step. Otherwise, it selects 1 from  $S_r(D) = \{0, 1\}$  with probability  $\frac{1}{2}$ . Therefore, by symmetry, (1) becomes

$$\text{RA}_F(T) = \sum_D \Pr_r[D|1](\Pr[S_r(D) = \{1\}] + \frac{1}{2} \Pr[S_r(D) = \{0, 1\}]). \quad (2)$$

Let

$$\Pr_X[S|s] = \sum_{D'} \Pr_X[D'|s] \Pr[S_X(D') = S]$$

for a node  $X$ , a state  $s \in \{0, 1\}$ ,  $S = \{1\}, \{0, 1\}$ , and a set  $D'$  of possible states of the leaves below  $X$ .  $\Pr_X[S|s]$  is the probability that the Fitch method outputs state subset  $S$  at  $X$  in its first step given the true state of  $X$  is  $s$ . By symmetry,

$$\begin{aligned} \Pr_X[\{1\}|1] &= \Pr_X[\{0\}|0], \\ \Pr_X[\{0\}|1] &= \Pr_X[\{1\}|0], \\ \Pr_X[\{0, 1\}|1] &= 1 - \Pr_X[\{1\}|1] - \Pr_X[\{0\}|1], \\ \Pr_X[\{0, 1\}|0] &= 1 - \Pr_X[\{1\}|0] - \Pr_X[\{0\}|0]. \end{aligned}$$

For a node  $X$  and a state  $s = 0, 1$ , we further set

$$\alpha_X = \Pr_X[\{s\}|s], \quad \beta_X = \Pr_X[\{1-s\}|s].$$

Then,

$$\Pr_X[\{0, 1\}|s] = 1 - \alpha_X - \beta_X.$$

Then, (2) becomes

$$\begin{aligned} \text{RA}_F(T) &= \Pr_r[\{1\}|1] + \frac{1}{2} \Pr_r[\{0, 1\}|1] \\ &= \frac{1}{2} + \frac{1}{2} (\Pr_r[\{1\}|1] - \Pr_r[\{0\}|1]) \\ &= \frac{1}{2} + \frac{1}{2} (\alpha_r - \beta_r). \end{aligned} \quad (3)$$

Let  $Z$  be an internal node and have  $X$  and  $Y$  as its children. Furthermore, we let the conservation probability on branches  $ZX$  and  $ZY$  be  $p_X$  and  $p_Y$ , respectively. The subset  $S_Z$  computed at  $Z$  is  $\{1\}$  if and only if one of  $S_X$  and  $S_Y$  is  $\{1\}$  and the other is  $\{1\}$  or  $\{0, 1\}$ . Hence,

$$\begin{aligned} \alpha_Z &= (p_X \alpha_X + q_X \beta_X)(p_Y \alpha_Y + q_Y \beta_Y) \\ &\quad + (p_X \alpha_X + q_X \beta_X)(1 - \alpha_Y - \beta_Y) \\ &\quad + (1 - \alpha_X - \beta_X)(p_Y \alpha_Y + q_Y \beta_Y), \end{aligned} \quad (4)$$

where  $q_X = 1 - p_X$  and  $q_Y = 1 - p_Y$ . Similarly

$$\begin{aligned}\beta_Z &= (q_X\alpha_X + p_X\beta_X)(q_Y\alpha_Y + p_Y\beta_Y) \\ &\quad + (q_X\alpha_X + p_X\beta_X)(1 - \alpha_Y - \beta_Y) \\ &\quad + (1 - \alpha_X - \beta_X)(q_Y\alpha_Y + p_Y\beta_Y)\end{aligned}\tag{5}$$

These two recurrence relations presented in [6] lead to an efficient dynamic programming method for calculating  $\alpha_r$  and  $\beta_r$ . But, these two relations are not simple enough for the theoretical study of the reconstruction accuracy. In the rest of this section, we shall establish two recurrence relations for the purpose of the theoretical analysis.

Let

$$C_Z = 1 - \alpha_Z - \beta_Z$$

and

$$D_Z = \alpha_Z - \beta_Z.$$

If  $Z$  is a leaf, we have that

$$C_Z = 0, \quad D_Z = 1.\tag{6}$$

Otherwise, we have the following recurrence relations.

**Theorem 3.1** *Let  $Z$  be an internal node and have children  $X$  and  $Y$ . Then,*

$$\begin{aligned}C_Z &= \frac{1}{2} \times [1 - C_X - C_Y + 3C_X C_Y \\ &\quad - (2p_X - 1)(2p_Y - 1)D_X D_Y],\end{aligned}\tag{7}$$

and

$$\begin{aligned}D_Z &= \frac{1}{2}(2p_X - 1)(1 + C_Y)D_X \\ &\quad + \frac{1}{2}(2p_Y - 1)(1 + C_X)D_Y.\end{aligned}\tag{8}$$

**Proof.** These two relations can be verified by using (4) and (5). The details can be found in Appendix.  $\square$

As the first application of this theorem, we obtain the following fact. This result can be found in [9]. Here we give a short proof.

**Corollary 3.1** *For any phylogenetic tree  $T$  with root  $r$  in which the conservation probability is at least  $\frac{1}{2}$ ,  $\Pr[0, 1|s] = C_r \leq \frac{1}{2}$  for  $s = 0, 1$ .*

**Proof.** We prove the fact by induction on  $n$ , the number of nodes of  $T$ . For  $n = 0$ , the fact follows from (6). Suppose  $C_r \leq \frac{1}{2}$  for any tree with less than  $n$  nodes. Now, consider a phylogenetic tree  $T$  of  $n$  nodes. Let the root  $r$  of  $T$  have children  $X$  and  $Y$ . Then, by induction,  $0 \leq C_X, C_Y \leq \frac{1}{2}$ . Since  $p_X, p_Y \geq 1/2$ , by Formula (7),

$$\begin{aligned}C_r &= (1/2)[2/3 + 3(C_X - 1/3)(C_Y - 1/3) \\ &\quad - (2p_X - 1)(2p_Y - 1)D_X D_Y] \\ &\leq (1/2)[2/3 + 3|C_X - 1/3| \times |C_Y - 1/3|] \\ &\leq (1/2)[2/3 + 3 \times (1/3)^2] \\ &= 1/2.\end{aligned}$$

Hence, the fact holds.  $\square$

## 4 Accuracy on complete binary trees

In this section, we study the reconstruction accuracy of the Fitch method on the complete binary trees. Let  $T_n$  be the complete binary tree of  $2^n$  leaves in which the conservation probability is  $p$  along each branch. Let  $r$  denote the root of  $T_n$  and  $C_n(p) = C_r$  and  $D_n(p) = D_r$  in  $T_n$ . Since the subtree rooted at each child of the root in  $T_n$  is the complete binary tree of  $2^{n-1}$  leaves, (7) and (8) imply that, for  $n \geq 1$ ,

$$\begin{aligned} 2C_n(p) &= 1 - 2C_{n-1}(p) + 3C_{n-1}^2(p) - (2p-1)^2 D_{n-1}^2(p), \\ D_n(p) &= (2p-1)(1 + C_{n-1}(p)) D_{n-1}(p), \end{aligned} \quad (9)$$

where  $0 \leq p \leq 1$ .

**Lemma 4.1** *For any  $n \geq 1$  and  $0 \leq p \leq 1$ ,*

$$C_n(p) = C_n(1-p), \quad |D_n(p)| = |D_n(1-p)|$$

.

**Proof.** We prove by induction on  $n$ . For  $n = 0$ , the facts follow from Formula (6).

Suppose now the lemma is true for  $n-1$ ; that is,  $C_{n-1}(p) = C_{n-1}(1-p)$  and  $|D_{n-1}(p)| = |D_{n-1}(1-p)|$ . Then

$$\begin{aligned} & 2C_n(p) \\ &= 1 - 2C_{n-1}(p) + 3C_{n-1}^2(p) - (2p-1)^2 D_{n-1}^2(p) \\ &= 1 - 2C_{n-1}(1-p) + 3C_{n-1}^2(1-p) \\ &\quad - (2(1-p)-1)^2 D_{n-1}^2(1-p) \\ &= 2C_n(1-p) \end{aligned}$$

and

$$\begin{aligned} & |D_n(p)| \\ &= |2p-1| \cdot (1 + C_{n-1}(p)) \cdot |D_{n-1}(p)| \\ &= |2(1-p)-1| \cdot (1 + C_{n-1}(1-p)) \cdot |D_{n-1}(1-p)| \\ &= |D_n(1-p)|, \end{aligned}$$

from which Lemma 4.1 follows by induction.  $\square$

By Lemma 4.1, we have

$$\lim_{n \rightarrow \infty} C_n(p) = \lim_{n \rightarrow \infty} C_n(1-p)$$

and

$$\lim_{n \rightarrow \infty} |D_n(p)| = \lim_{n \rightarrow \infty} |D_n(1-p)|,$$

if all the above limits exist. Therefore, it suffices to assume that  $1/2 \leq p \leq 1$ . Now we simplify our notations by dropping  $p$  from two equalities in (9), resulting in

$$2C_n = 1 - 2C_{n-1} + 3C_{n-1}^2 - (2p-1)^2 D_{n-1}^2, \quad (10)$$

$$D_n = (2p-1)(1 + C_{n-1})D_{n-1}. \quad (11)$$

**Lemma 4.2** For any  $n \geq 1$ ,

$$0 \leq C_n \leq \frac{1}{2}, \quad 0 \leq D_n \leq 1.$$

**Proof.** Since we assume  $1/2 \leq p \leq 1$ , the first fact is from Corollary 3.1. The second fact is trivial.  $\square$

**Lemma 4.3** Let  $n \geq 1$ . If  $C_{n-1} \leq \frac{1}{3}$ , then  $C_n \leq \frac{1}{3}$ .

**Proof.** We rewrite Formula (10) as

$$2 \left( \frac{1}{3} - C_n \right) + 3 \left( \frac{1}{3} - C_{n-1} \right)^2 - (2p-1)^2 D_{n-1}^2 = 0. \quad (12)$$

This implies that

$$0 \leq 2 \left( \frac{1}{3} - C_{n-1} \right) \leq (2p-1)^2 D_{n-2}^2,$$

and

$$4 \left( \frac{1}{3} - C_{n-1} \right)^2 \leq (2p-1)^4 D_{n-2}^4.$$

By Lemma 4.2, we have that

$$\begin{aligned} 2C_n &= \frac{2}{3} + 3 \left( \frac{1}{3} - C_{n-1} \right)^2 - (2p-1)^2 D_{n-1}^2 \\ &\leq \frac{2}{3} + \frac{3}{4} (2p-1)^4 D_{n-2}^4 \\ &\quad - (2p-1)^2 [(2p-1)(1+C_{n-2})D_{n-2}]^2 \\ &= \frac{2}{3} + (2p-1)^4 \left( \frac{3}{4} D_{n-2}^2 - (1+C_{n-2})^2 \right) D_{n-2}^2 \\ &\leq \frac{2}{3} + (2p-1)^4 \left( \frac{3}{4} - (1+0)^2 \right) D_{n-2}^2 \\ &\leq \frac{2}{3}. \end{aligned}$$

and hence Lemma 4.3 follows.  $\square$

**Lemma 4.4** Let  $n \geq 1$ . If  $C_{n-1} \geq \frac{1}{3}$ , then  $C_n \leq C_{n-1}$ .

**Proof.**

$$\begin{aligned} 2C_n &= 1 - 2C_{n-1} + 3C_{n-1}^2 - (2p-1)^2 D_{n-1}^2 \\ &= 2C_{n-1} + (1 - C_{n-1})(1 - 3C_{n-1}) - (2p-1)^2 D_{n-1}^2 \\ &\leq 2C_{n-1}. \end{aligned}$$

$\square$

**Theorem 4.1** Suppose  $\frac{1}{8} \leq p < \frac{7}{8}$ . Then

$$\lim_{n \rightarrow \infty} C_n = \frac{1}{3}, \quad \lim_{n \rightarrow \infty} D_n = 0.$$

**Proof.** The proof is divided into two cases.

Case 1:  $C_n \geq 1/3$  for all  $n$ . By Lemma 4.4,  $C_n$  is a decreasing positive sequence and thus  $\lim_{n \rightarrow \infty} C_n$  exists and its value is at least  $1/3$ . The equality  $2C_n = 1 - 2C_{n-1} + 3C_{n-1}^2 - (2p-1)^2 D_{n-1}^2$  implies that  $\lim_{n \rightarrow \infty} D_n$  exists. Taking limits on all terms in (11) implies that  $\lim_{n \rightarrow \infty} D_n = 0$  since  $\lim_{n \rightarrow \infty} C_n \geq 1/3$ . Again, taking on all terms in (10) gives that

$$2 \lim_{n \rightarrow \infty} C_n = 1 - 2 \lim_{n \rightarrow \infty} C_n + 3 \left( \lim_{n \rightarrow \infty} C_n \right)^2 - 0;$$

that is,  $\lim_{n \rightarrow \infty} C_n = 1/3$  or 1. Since  $C_n$  is decreasing and  $C_1 = 2p(1-p) < 1/2$ ,  $\lim_{n \rightarrow \infty} C_n \neq 1$ . Thus  $\lim_{n \rightarrow \infty} C_n = 1/3$ .

Case 2:  $C_N < 1/3$  for some  $N$ . By Lemma 4.3,  $C_n \leq 1/3$  for all  $n \geq N$ . Formula (11) implies that

$$D_n = (2p-1)(1+C_{n-1})D_{n-1} \leq \left(\frac{4}{3}(2p-1)\right)^{n-N} D_{N-1}$$

for any  $n \geq N$ . Since  $1/2 \leq p < 7/8$ ,  $\frac{4}{3}(2p-1) < 1$  and hence  $\lim_{n \rightarrow \infty} D_n = 0$ .

By Formula (12),

$$2\left(\frac{1}{3} - C_n\right) = (2p-1)^2 D_{n-1}^2 - 3\left(C_{n-1} - \frac{1}{3}\right)^2$$

and hence

$$2\left(\frac{1}{3} - C_n\right) \leq (2p-1)^2 D_{n-1}^2$$

for all  $n \geq N$ . Since

$$0 \leq 2\left(\frac{1}{3} - C_n\right)$$

and

$$\lim_{n \rightarrow \infty} (2p-1)^2 D_{n-1}^2 = (2p-1)^2 \left(\lim_{n \rightarrow \infty} D_{n-1}\right)^2 = 0,$$

by the Sandwich Theorem

$$\lim_{n \rightarrow \infty} 2\left(\frac{1}{3} - C_n\right) = 0$$

and thus  $\lim_{n \rightarrow \infty} C_n = 1/3$ . □

To prove the convergence of  $C_n$  and  $D_n$  for  $p \geq \frac{7}{8}$ , we set

$$c_n = 2(1-p)/(2p-1) - C_n$$

and

$$d_n = D_n^2.$$

Then, Formula (12) implies that

$$\begin{aligned} & 2\left(\frac{2(1-p)}{2p-1} - c_n\right) \\ &= \frac{2}{3} + 3\left(\frac{1}{3} - \frac{2(1-p)}{2p-1} + c_{n-1}\right)^2 - (2p-1)^2 d_{n-1} \\ &= \frac{2}{3} + 3\left(\frac{8p-7}{3(2p-1)} + c_{n-1}\right)^2 - (2p-1)^2 d_{n-1} \\ &= \frac{2}{3} + \frac{(8p-7)^2}{3(2p-1)^2} + \frac{2(8p-7)}{2p-1} c_{n-1} + 3c_{n-1}^2 - (2p-1)^2 d_{n-1}, \end{aligned}$$

or equivalently

$$\begin{aligned} 2c_n &= (2p-1)^2 d_{n-1} - \frac{2(8p-7)}{2p-1} c_{n-1} - 3c_{n-1}^2 \\ &\quad - \frac{(8p-7)(4p-3)}{(2p-1)^2}. \end{aligned} \tag{13}$$

Formula (11) implies that

$$\begin{aligned} d_n &= (2p-1)^2 \left( \frac{1}{2p-1} - c_n \right)^2 d_{n-1} \\ &= [1 - (2p-1)c_{n-1}]^2 d_{n-1}. \end{aligned} \quad (14)$$

**Lemma 4.5** For any  $k \geq 2$  and  $p \geq 7/8$ ,

- (1)  $c_k \geq 0$ .
- (2)  $d_{k+1} \leq d_k$ .
- (3)  $c_k \leq \frac{5(1-p)}{4(2p-1)}$ .

**Proof.** We prove it by induction on  $k$ . The facts is obviously true for  $k = 2, 3$ . Assume they hold for  $k \leq n-1$ . We now prove they hold for  $k = n$ .

- (1). By induction,  $0 \leq c_{n-2}, c_{n-1} \leq \frac{5(1-p)}{4(2p-1)}$ . Hence,

$$\begin{aligned} & [1 - (2p-1)c_{n-2}]^2 - \frac{8p-7}{2p-1} - \frac{3}{2}c_{n-1} \\ &= \frac{6(1-p)}{2p-1} - 2(2p-1)c_{n-2} - \frac{3}{2}c_{n-1} + (2p-1)^2 c_{n-2}^2 \\ &\geq \frac{6(1-p)}{2p-1} - \frac{8p-1}{2} \times \frac{5(1-p)}{4(2p-1)} + 0 \\ &= \frac{1-p}{2p-1} \times \frac{53-40p}{8} \\ &\geq 0. \end{aligned} \quad (15)$$

Setting  $\Delta = \frac{(8p-7)(4p-3)}{(2p-1)^2}$ , we have

$$\begin{aligned} & 2c_n \\ &= (2p-1)^2 d_{n-1} - \frac{2(8p-7)}{2p-1} c_{n-1} - 3c_{n-1}^2 - \Delta \\ &= (2p-1)^2 d_{n-1} - 2c_{n-1} \left( \frac{8p-7}{2p-1} + \frac{3}{2}c_{n-1} \right) - \Delta. \end{aligned}$$

By using recurrence (13) and (14), we obtain that

$$\begin{aligned} & 2c_n \\ &= (2p-1)^2 (1 - (2p-1)c_{n-2})^2 d_{n-2} - \left[ \frac{(8p-7)}{2p-1} + \frac{3}{2}c_{n-1} \right] \\ & \quad \times \left[ (2p-1)^2 d_{n-2} - \frac{2(8p-7)}{2p-1} c_{n-2} - 3c_{n-2}^2 - \Delta \right] - \Delta \\ &= (2p-1)^2 \left[ (1 - (2p-1)c_{n-2})^2 - \frac{8p-7}{2p-1} - \frac{3}{2}c_{n-1} \right] d_{n-2} \\ & \quad + \left[ \frac{8p-7}{2p-1} + \frac{3}{2}c_{n-1} \right] \left[ \frac{2(8p-7)}{2p-1} c_{n-2} + 3c_{n-2}^2 + \Delta \right] - \Delta. \end{aligned}$$

Since  $c_{n-1} \geq 0$ , Formula (13) implies that

$$(2p-1)^2 d_{n-2} \geq \frac{2(8p-7)}{2p-1} c_{n-2} + 3c_{n-2}^2 + \Delta.$$



This inequality and (15) implies that

$$\begin{aligned}
& 2c_n \\
& \geq [(1 - (2p - 1)c_{n-2})^2 - \frac{8p-7}{2p-1} - \frac{3}{2}c_{n-1}] \\
& \quad \times [\frac{2(8p-7)}{2p-1}c_{n-2} + 3c_{n-2}^2 + \Delta] \\
& \quad + [\frac{8p-7}{2p-1} + \frac{3}{2}c_{n-1}][\frac{2(8p-7)}{2p-1}c_{n-2} + 3c_{n-2}^2 + \Delta] - \Delta \\
& = \frac{8(8p-7)(1-p)}{2p-1}c_{n-2} + [3 + (8p-7)(4p-7)]c_{n-2}^2 \\
& \quad + 4(2p-1)(4p-5)c_{n-2}^3 + 3(2p-1)^2c_{n-2}^4.
\end{aligned}$$

By assumption,  $c_{n-2} \leq \frac{5(1-p)}{4(2p-1)}$  and  $4p-5 < -1$ . Replacing  $c_{n-2}^3$  with  $\frac{5(1-p)}{4(2p-1)}c_{n-2}^2$  in the right-hand side of the last inequality, we have that

$$\begin{aligned}
& 2c_n \\
& \geq \frac{8(8p-7)(1-p)}{2p-1}c_{n-2} + [3 + (8p-7)(4p-7)]c_{n-2}^2 \\
& \quad + 5(1-p)(4p-5)c_{n-2}^2 + 3(2p-1)^2c_{n-2}^4 \\
& = \frac{8(8p-7)(1-p)}{2p-1}c_{n-2} + 3(1-p)(9-4p)c_{n-2}^2 \\
& \quad + 3(2p-1)^2c_{n-2}^4 \\
& \geq 0
\end{aligned}$$

(2) We have proved that  $c_n \geq 0$ . Therefore,

$$d_{n+1} = [1 - (2p-1)c_n]^2 d_n \leq d_n.$$

(3) Since  $d_k$  decreases for  $k \leq n$ ,

$$d_n \leq d_2 = D_2^2 < (2p-1)^2 \quad (16)$$

Let  $q = 1 - p$ . Note that  $p \geq \frac{7}{8}$  and  $q \leq \frac{1}{8}$ . Therefore, we have that

$$\frac{1}{1-2q} \leq \frac{4}{3}$$

and

$$16q(1-5q) \leq 16 \times \frac{1}{10} \times \left(1 - 5 \times \frac{1}{10}\right) = \frac{4}{5}.$$

Recalling that  $c_{n-1} \geq 0$ , by (16), we have that

$$\begin{aligned}
& c_n \\
& = \frac{1}{2}[(2p-1)^2 d_{n-1} - \frac{2(8p-7)}{2p-1}c_{n-1} - 3c_{n-1}^2 \\
& \quad - \frac{(8p-7)(4p-3)}{(2p-1)^2}] \\
& \leq \frac{1}{2}[(2p-1)^2 d_{n-1} - \frac{(8p-7)(4p-3)}{(2p-1)^2}] \\
& = \frac{1}{2(2p-1)^2}[(2p-1)^4 d_{n-1} - (8p-7)(4p-3)] \\
& \leq \frac{1}{2(2p-1)^2}[(2p-1)^6 - (8p-7)(4p-3)] \\
& = \frac{1}{2(2p-1)^2}[(1-2q)^6 - (1-8q)(1-4q)] \\
& = \frac{q}{(2p-1)^2}[2q(7-40q+60q^2-48q^3+16q^4)] \\
& \leq \frac{q}{(2p-1)^2}[2q(7-40q+60q^2+16q^4)] \\
& \leq \frac{q}{(2p-1)^2}[2q(7-40q+\frac{60}{64}+\frac{1}{256})] \\
& \leq \frac{q}{(2p-1)^2}[2q(8-40q)] \\
& = \frac{q}{2p-1} \frac{16q(1-5q)}{1-2q} \\
& \leq \frac{4q}{5(2p-1)} \frac{1}{1-2q}
\end{aligned}$$

Since  $q \leq \frac{1}{8}$  and  $\frac{1}{1-2q} \leq \frac{4}{3}$ ,  $c_n \leq \frac{16q}{15(2p-1)} \leq \frac{5q}{4(2p-1)}$ .  $\square$

**Theorem 4.2** Suppose  $\frac{7}{8} \leq p \leq 1$ . Then

$$\lim_{n \rightarrow \infty} C_n = \frac{2(1-p)}{2p-1}$$

and

$$\lim_{n \rightarrow \infty} D_n^2 = \frac{(8p-7)(4p-3)}{(2p-1)^4}.$$

**Proof.** Since  $c_n \geq 0$  for all  $n$ , Formula (13) implies

$$d_n \geq \frac{(8p-7)(4p-3)}{(2p-1)^4}$$

for all  $n$ . Since  $d_n = D_n^2$  is a decreasing sequence,  $\lim_{n \rightarrow \infty} d_n$  exists and is at least  $\frac{(8p-7)(4p-3)}{(2p-1)^4}$ , which is larger than 0 for  $p > \frac{7}{8}$ . Since  $0 \leq c_n \leq 1$ ,

$$0 \leq 1 - (2p-1)c_n \leq 1.$$

For  $p > \frac{7}{8}$ , Formula (14) implies that

$$\lim_{n \rightarrow \infty} 1 - (2p-1)c_n = 1$$

and so

$$\lim_{n \rightarrow \infty} c_n = 0.$$

Hence,  $\lim_{n \rightarrow \infty} C_n = \frac{2(1-p)}{2p-1}$ .

For  $p = \frac{7}{8}$ , Formulas (13) and (14) become

$$2c_n + 3c_{n-1}^2 = \frac{9}{16}d_{n-1}$$

and

$$d_n = \left(1 - \frac{3}{4}c_{n-1}\right)^2 d_{n-1}.$$

As a decreasing sequence,  $d_n$  has a non-negative limit. If  $\lim_{n \rightarrow \infty} d_n = 0$ , by the Sandwich theorem,  $\lim_{n \rightarrow \infty} c_n = 0$  from the fact that  $0 \leq 2c_n \leq \frac{9}{16}d_{n-1}$ . Therefore,

$$\lim_{n \rightarrow \infty} C_n = \frac{2(1-p)}{2p-1}$$

and

$$\lim_{n \rightarrow \infty} D_n^2 = \frac{(8p-7)(4p-3)}{(2p-1)^4}.$$

If  $\lim_{n \rightarrow \infty} d_n > 0$ , then,

$$d_n = d_{n-1} \left(1 - \frac{3}{4}c_{n-1}\right)^2$$

implies that  $\lim_{n \rightarrow \infty} c_n = 0$  and hence  $\lim_{n \rightarrow \infty} d_n = 0$ , a contradiction.  $\square$

**Theorem 4.3** *Let  $T_n$  be the complete binary tree of  $2^n$  leaves in which the conservation rate is  $p$  along each branch. In the two-state Jukes-Cantor model,*

- (a) (Steel [9]) *the accuracy of the Fitch method for reconstructing the root state in  $T_n$  converges as  $n$  goes to infinity to  $\frac{1}{2} + \frac{1}{2(2p-1)^2} \sqrt{(8p-7)(4p-3)}$  if  $p \in [\frac{7}{8}, 1]$  and  $\frac{1}{2}$  if  $p \in [\frac{1}{8}, \frac{7}{8}]$ .*
- (b) *it diverges as  $n$  goes to infinity if  $p \in (0, \frac{1}{8})$ ;*

**Proof.** By Formula (3) and the definition of  $D_n$ ,

$$RA_F(T_n) = \frac{1}{2} + \frac{1}{2}D_n.$$

Hence, the fact (a) follows from Theorems 4.1 and 4.2.

When  $0 < p < \frac{1}{8}$ ,  $D_n > 0$  for even integers  $n$  and  $D_n < 0$  for odd integers  $n$ . By Lemma 4.1 and Theorem 4.2,  $|D_n|$  converges to a positive number. Hence  $D_n$  and  $RA_F(T_n)$  diverge.  $\square$

## 5 The reconstruction accuracy on ultrametric trees

We now consider the accuracy of reconstructing the root state in ultrametric phylogenies. In an ultrametric phylogeny  $T$ , a branch  $xy$  has a length  $t_{xy}$ , but all the leaves have the same distance from the root. Under the two-state Jukes-Cantor model, the conservation probability  $p_{xy}$  along a branch  $xy$  of length  $t_{xy}$  is

$$p_{xy} = \frac{1}{2}(1 + e^{-2\lambda t_{xy}}),$$

where  $\lambda$  is a constant, representing the substitution rate in  $T$ . For an internal node  $u$  of  $T$ , the distance between it and any of its leaf descendants is defined as its depth, denoted by  $d(u)$ .

**Lemma 5.1** *Let  $T$  be an ultrametric phylogeny and  $u$  an internal node. Under the 2-state Jukes-Cantor model, for any path  $P(x, y)$  from an internal node  $x$  to its leaf descendant  $y$ ,*

$$\prod_{uv \in P(x, y)} (2p_{uv} - 1) = e^{-2\lambda d(x)}. \quad (17)$$

**Proof.** It follows from that  $2p_{uv} - 1 = e^{-2\lambda t_{uv}}$  for each edge  $uv$  and that  $d(x) = \sum_{uv \in P(x, y)} t_{uv}$ .  $\square$

Let  $T$  be an ultrametric tree that has three or more leaves. For any internal node  $w$  with children  $w_1$  and  $w_2$ , by Formula (8) and Lemma 5.1, we have that

$$D_w \geq \frac{1}{2}(2p_{ww_1} - 1)D_{w_1} + \frac{1}{2}(2p_{ww_2} - 1)D_{w_2} \quad (18)$$

because  $C_{w_1}, C_{w_2} \geq 0$ . By induction, we can show the following fact from Formula (18).

**Lemma 5.2**

$$D_w \geq \prod_{(u, v) \in P(w, l)} (2p_{uv} - 1) = e^{-2\lambda d(w)},$$

where  $l$  is a leaf below  $w$ .

By Formula (3), the above lemma implies that the accuracy of reconstructing the root state from all the leaf states is not less than from a single leaf. Such a fact was established by Fischer and Thatte in [2]. It can be strengthened as follows.

**Theorem 5.1** *Let  $T$  be an ultrametric tree having three or more leaves and let  $x$  be a child of its root  $r$ . If  $x$  has two children, then*

$$D_r \geq e^{-2\lambda d(r)} \left[ 1 + \frac{1}{4}(1 - e^{-4\lambda d(x)}) \right] \quad (19)$$

**Proof.** By Lemma 5.2,  $D_y \geq e^{-2\lambda d(y)}$ . Since  $C_y \geq 0$ , by Formula (8), we have that

$$\begin{aligned} D_r &= \frac{1}{2}(2p_{rx} - 1)(1 + C_y)D_x + \frac{1}{2}(2p_{ry} - 1)(1 + C_x)D_y \\ &\geq \frac{1}{2}(2p_{rx} - 1)D_x + \frac{1}{2}e^{-2\lambda d(r)}(1 + C_x) \end{aligned} \quad (20)$$

Let  $u$  and  $v$  be the children of  $x$ . By Lemma 5.2,  $D_u \geq e^{-2\lambda d(u)}$  and  $D_v \geq e^{-2\lambda d(v)}$ . Let

$$D_u = e^{-2\lambda d(u)}(1 + \Delta(u)), \quad D_v = e^{-2\lambda d(v)}(1 + \Delta(v)),$$

where  $\Delta(u), \Delta(v) \geq 0$ . We then have

$$\begin{aligned} D_x &= \frac{1}{2}(2p_{xu} - 1)(1 + C_v)D_u + \frac{1}{2}(2p_{xv} - 1)(1 + C_u)D_v \\ &= e^{-2\lambda d(x)} \left\{ \frac{1}{2}[1 + C_v + \Delta(u) + C_v\Delta(u)] \right. \\ &\quad \left. + \frac{1}{2}[1 + C_u + \Delta(v) + C_u\Delta(v)] \right\} \\ &\geq e^{-2\lambda d(x)} \left\{ 1 + \frac{1}{2}[C_u + C_v + \Delta(u) + \Delta(v)] \right\}. \end{aligned} \quad (21)$$

Combining Formulas (20) and (21) gives that

$$D_r \geq e^{-2\lambda d(r)} \left\{ 1 + \frac{1}{2}C_x + \frac{1}{4}[C_u + C_v + \Delta(u) + \Delta(v)] \right\}$$

By Formula (7),

$$\begin{aligned} C_x &= \frac{1}{2}[1 - C_u - C_v + 3C_uC_v - (2p_{xu} - 1)(2p_{xv} - 1)D_uD_v] \\ &\geq \frac{1}{2}[1 - C_u - C_v - (2p_{xu} - 1)(2p_{xv} - 1)D_uD_v], \\ &= \frac{1}{2}\{1 - C_u - C_v - e^{-4\lambda d(x)}[1 + \Delta(u)][1 + \Delta(v)]\}. \end{aligned}$$

We further have that

$$\begin{aligned} D_r &\geq \frac{1}{4}e^{-2\lambda d(r)} \\ &\quad \times \{5 + \Delta(u) + \Delta(v) - e^{-4\lambda d(x)}[1 + \Delta(u)][1 + \Delta(v)]\}. \end{aligned}$$

Since  $d(x) > d(v)$ ,

$$[1 + \Delta(v)]e^{-4\lambda d(x)} \leq [1 + \Delta(v)]e^{-2\lambda d(v)} = D_v \leq 1.$$

Therefore, we obtain that

$$\begin{aligned} & \Delta(u) + \Delta(v) - e^{-4\lambda d(x)}[1 + \Delta(u)][1 + \Delta(v)] \\ & \geq \Delta(u) - e^{-4\lambda d(x)}[1 + \Delta(u)] \\ & \geq -e^{-4\lambda d(x)} \end{aligned}$$

and

$$D_r \geq \frac{1}{4}e^{-2\lambda d(r)}(5 - e^{-4\lambda d(x)}) = e^{-2\lambda d(r)}[1 + \frac{1}{4}(1 - e^{-4\lambda d(x)})].$$

□

It is known that there exists an ultrametric tree in which the root state can be reconstructed more accurately from the states of a subset of four leaves than from all the leaf states. Let  $l_1, l_2, l_3$  be three leaves in  $T$ . Assume that the least common ancestor (lca)  $t$  of  $l_2$  and  $l_3$  is not the root  $r$  and has depth  $d(t)$ . If the lca of  $l_1$  and  $t$  is the root, then, the accuracy of reconstructing the root state from these three leaves is  $\frac{1}{2} + \frac{1}{2}e^{-2\lambda d(r)}[1 + \frac{1}{4}(1 - e^{-4\lambda d(t)})]$ , which is at most  $\frac{1}{2} + \frac{1}{2}e^{-2\lambda d(r)}[1 + \frac{1}{4}(1 - e^{-4\lambda d(x)})]$  because  $d(x) \geq d(t)$ . If the lca of  $l_1$  and  $t$  is not  $r$ , the accuracy is even smaller. Therefore, Theorem 5.1 implies that the reconstruction of the root state from all the leaf states is at least as accurately as from the states of any three leaves.

## References

- [1] M.L. Baba, M. Goodman, J. Berger-Cohn, J.G. Demaille, G. Matsuda. 1984. The early adaptive evolution of calmodulin. *Mol. Biol. Evol.* 1: 442-455.
- [2] M Fischer, BD Thatte. 2008. Maximum Parsimony on Subsets of Taxa, *J. Theoret. Biol.* (accepted).
- [3] Fitch, W. M. 1971. Toward Defining the Course of Evolution: Minimum change for a specific tree topology. *Syst. Zool.* 20:406-416.
- [4] G.L. Li, M. Steel and L.X. Zhang. 2008. More taxa are not necessarily better for the reconstruction of ancestral character states. *Syst. Biol.* 57:647-653.
- [5] Liberles, D.A. (ed.). 2007, Ancestral sequence reconstruction, Oxford University Press, USA
- [6] Maddison, W. P. 1995. Calculating the probability distributions of ancestral states reconstructed by parsimony on phylogenetic trees. *Syst. Biol.* 44:474-481.
- [7] L. Pauling and E. Zuckerkandl, 1963. Chemical paleogenetics: molecular restoration studies of extinct forms of lives. *Acta Chem. Scand.* 17: S9-S16.
- [8] Salisbury, B.A. and J. Kim. 2001. Ancestral state estimation and taxon sampling density, *Syst. Biol.* 50:557-564.
- [9] M. Steel. 1989. Distribution in bicoloured evolutionary trees. *PhD thesis*, Massey University, New Zealand.
- [10] J.W. Thornton, 2004. Resurrecting ancient genes: Experimental analysis of extinct molecules. *Nature Review Genetics* 5: 366-375.
- [11] J. Zhang and M. Nei, 1997. Accuracies of ancestral amino acid sequences inferred by parsimony, likelihood, and distance methods. *J. Mol. Evol.* 44:139-146.

### Appendix: Proof of Theorem 3.1

We first have that

$$(p_X\alpha_X + q_X\beta_X)(1 - \alpha_Y - \beta_Y) - (q_X\alpha_X + p_X\beta_X)(1 - \alpha_Y - \beta_Y) = (2p_X - 1)C_Y D_X, \quad (22)$$

$$(1 - \alpha_X - \beta_X)(p_Y\alpha_Y + q_Y\beta_Y) - (1 - \alpha_X - \beta_X)(q_Y\alpha_Y + p_Y\beta_Y) = (2p_Y - 1)C_X D_Y, \quad (23)$$

and

$$\begin{aligned} & (p_X\alpha_X + q_X\beta_X)(p_Y\alpha_Y + q_Y\beta_Y) - (q_X\alpha_X + p_X\beta_X)(q_Y\alpha_Y + p_Y\beta_Y) \\ &= (p_X + p_Y - 1)(\alpha_X\alpha_Y - \beta_X\beta_Y) + (b - a)(\beta_X\alpha_Y - \alpha_X\beta_Y). \end{aligned} \quad (24)$$

Since

$$\alpha_X\alpha_Y - \beta_X\beta_Y = (\alpha_X - \beta_X)\alpha_Y + \beta_X(\alpha_Y - \beta_Y)$$

and

$$\beta_X\alpha_Y - \alpha_X\beta_Y = \alpha_X(\alpha_Y - \beta_Y) - (\alpha_X - \beta_Y)\alpha_Y,$$

combining the equalities (22)-(24) given above leads to

$$D_Z = (2p_X - 1)(1 - \beta_Y)D_X + (2p_Y - 1)(1 - \alpha_X)D_Y.$$

By symmetry,

$$D_Z = (2p_X - 1)(1 - \alpha_Y)D_X + (2p_Y - 1)(1 - \beta_X)D_Y.$$

Therefore,

$$\begin{aligned} D_Z &= \frac{1}{2}(2p_X - 1)(2 - \alpha_Y - \beta_Y)D_X + \frac{1}{2}(2p_Y - 1)(2 - \alpha_X - \beta_X)D_Y \\ &= \frac{1}{2}(2p_X - 1)(1 + C_Y)D_X + \frac{1}{2}(2p_Y - 1)(1 + C_X)D_Y \end{aligned} \quad (25)$$

Moreover, we also have that

$$\begin{aligned} & \alpha_Z + \beta_Z \\ &= (p_X p_Y + q_X q_Y)(\alpha_X \alpha_Y + \beta_X \beta_Y) + (q_X p_Y + p_X q_Y)(\beta_X \alpha_Y + \alpha_X \beta_Y) \\ &+ (\alpha_X + \beta_X)(1 - \alpha_Y - \beta_Y) + (1 - \alpha_X - \alpha_Y)(\alpha_Y + \beta_Y). \end{aligned}$$

Since

$$\alpha_X \alpha_Y + \beta_X \beta_Y = \frac{1}{2}((1 - C_X)(1 - C_Y) + D_X D_Y)$$

and

$$\beta_X \alpha_Y + \alpha_X \beta_Y = \frac{1}{2}((1 - C_X)(1 - C_Y) - D_X D_Y),$$

we obtain that

$$1 - C_Z = \frac{1}{2}[1 + C_X + C_Y - 3C_X C_Y + (2p_X - 1)(2p_Y - 1)D_X D_Y],$$

or equivalently

$$C_Z = \frac{1}{2}[1 - C_X - C_Y + 3C_X C_Y - (2p_X - 1)(2p_Y - 1)D_X D_Y]. \quad (26)$$